

# On the Schrödinger equation in $\mathbb{R}^N$ under the effect of a general nonlinear term

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## Abstract

In this paper we prove the existence of a positive solution to the equation  $-\Delta u + V(x)u = g(u)$  in  $\mathbb{R}^N$ , assuming the general hypotheses on the nonlinearity introduced by Berestycki & Lions. Moreover we show that a minimizing problem, related to the existence of a ground state, has no solution.

## 1 Introduction

This paper deals with the following equation:

$$\begin{cases} -\Delta u + V(x)u = g(u), & x \in \mathbb{R}^N, \quad N \geq 3; \\ u > 0. \end{cases} \quad (1)$$

An existence result of nontrivial solutions for this kind of problem has been obtained by Rabinowitz [8] assuming that  $g$  is superlinear and subcritical at infinity and satisfies the global growth Ambrosetti-Rabinowitz condition

$$\exists \mu > 2 \text{ s.t. } 0 < \mu \int_0^t g(s) ds \leq g(t)t, \text{ for all } t \in \mathbb{R}. \quad (2)$$

This condition is used to get the boundedness of Palais-Smale sequences. In [7], L. Jeanjean and K. Tanaka have been able to remove the hypothesis

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(2) by means of an abstract tool, consisting in a suitable approximating method (see [4, Theorem 1.1]). However they preserved the condition on the superlinear growth at infinity.

On the other hand, in the fundamental paper [2], Berestycki & Lions proved the existence of a ground state, namely a solution which minimizes the action among all the solutions, for the problem

$$\begin{cases} -\Delta u = g(u), & \text{in } \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), u \neq 0, \end{cases} \quad (3)$$

under the following assumptions on the nonlinearity  $g$ :

(g1)  $g \in C(\mathbb{R}^N, \mathbb{R})$ ,  $g$  odd;

(g2)  $-\infty < \liminf_{s \rightarrow 0^+} g(s)/s \leq \limsup_{s \rightarrow 0^+} g(s)/s = -m < 0$ ;

(g3)  $-\infty < \limsup_{s \rightarrow +\infty} g(s)/s^{2^*-1} \leq 0$ ;

(g4) there exists  $\zeta > 0$  such that  $G(\zeta) := \int_0^\zeta g(s) ds > 0$ ;

here  $2^* = 2N/(N-2)$ .

So it seems that the “natural” assumptions on the nonlinearity do not require the superlinearity at infinity.

In view of this result, the aim of this paper is to study the problem (1) preserving the same general assumptions of [2] on  $g$ . Moreover we assume the following hypotheses on  $V$  :

(V1)  $V \in C^1(\mathbb{R}^N, \mathbb{R})$  and  $V(x) \geq 0$ , for all  $x \in \mathbb{R}^N$ , and the inequality is strict somewhere;

(V2)  $\|(\nabla V(\cdot) \mid \cdot)^+\|_{N/2} < 2S$ ;

(V3)  $\lim_{|x| \rightarrow \infty} V(x) = 0$ ;

(V4)  $V$  is radially symmetric;

here  $(\nabla V(x) \mid x)^+ = \max\{(\nabla V(x) \mid x), 0\}$  and  $S$  is the best Sobolev constant of the embedding  $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ , namely

$$S = \inf_{u \in \mathcal{D}^{1,2} \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.$$

Our first main result is the following

**Theorem 1.1.** *Assume that (g1-4) and (V1-4) hold, then the problem (1) possesses at least a radially symmetric solution.*

Up to our knowledge, this is the first result on a problem as (1), where exactly the same general hypotheses of Berestycki & Lions [2] are assumed on the nonlinearity  $g$ .

As a consequence of Theorem 1.1, we can prove the following

**Theorem 1.2.** *Assuming that (g1-4) and (V1-4) hold, then the problem (1) possesses a radial ground state solution, namely a solution minimizing the action among all the nontrivial radial solutions.*

**Remark 1.3.** *We can generalize the hypotheses (V1) and (V3), requiring:*

(V1)'  $V \in C^1(\mathbb{R}^N, \mathbb{R})$  and  $V(x) \geq V_0 > -m$ , for all  $x \in \mathbb{R}^N$ , and the inequality is strict somewhere;

(V3)'  $\lim_{|x| \rightarrow \infty} V(x) = V_0$ ;

and supposing that the function  $\tilde{g}(s) = g(s) - V_0 s$  satisfies (g1-4).

**Remark 1.4.** *The geometrical hypotheses on the potential  $V$  do not allow us to use concentration-compactness arguments as in [7]. As a consequence, we have to require a symmetry property on  $V$  to prevent any possible loss of mass at infinity.*

In the second part of the paper, we are interested in solving a minimization problem strictly related to the existence of a ground state solution for (1). Let

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - \int_{\mathbb{R}^N} G(u), \quad u \in H^1(\mathbb{R}^N).$$

When the nonlinearity  $g$  satisfies (2), a standard method to look for the existence of a ground state solution for an equation as (1) is to study the minimizing problem

$$I(\bar{u}) = \inf_{u \in \mathcal{N}} I(u), \quad \bar{u} \in \mathcal{N}, \quad (4)$$

where  $\mathcal{N}$  is the Nehari manifold related to  $I$ . In [8] the geometrical assumption on  $V$

$$V(y) \leq \lim_{x \rightarrow \infty} V(x), \quad \text{for all } y \in \mathbb{R}^N \quad (5)$$

is used to solve such a minimizing problem. Moreover, in [7] it has been proved that, assuming (5), there exists a ground state solution for (1) also without the condition (2). On the other hand, it is well known (see for example [1]) that if  $g$  satisfies (2), and (5) holds with the reverse inequality, the minimizing problem (4) cannot be solved. In fact, a contradiction

argument deriving from the comparison of the level  $\eta := \inf_{u \in \mathcal{N}} I(u)$  with  $\eta_0 := \inf_{u \in \mathcal{N}_0} I_0(u)$  (here  $I_0$  and  $\mathcal{N}_0$  are the functional and the Nehari manifold of the problem at infinity) is used and the Ambrosetti-Rabinowitz condition plays a fundamental role. Actually, when we do not assume such a type of growth condition on  $g$ , these arguments do not work any more. However, a similar study can be done by replacing the Nehari manifold with a more suitable one.

Indeed, if we define by

$$\mathcal{P}_0 := \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 = N \int_{\mathbb{R}^N} G(u) \right\}, \quad (6)$$

the Pohozaev manifold related to (3), and we set

$$\mathcal{S}_0 := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid u \text{ is a solution of (3)}\},$$

it is well known that  $\mathcal{S}_0 \subset \mathcal{P}_0$ . Moreover, in [9] it has been proved that  $\mathcal{P}_0$  is a natural constraint for the functional related to (3)

$$I_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u), \quad u \in H^1(\mathbb{R}^N).$$

In order to look for a ground state solution, a very natural question arises: is the infimum  $I_0|_{\mathcal{P}_0}$  achieved?

Following two different ways, Jeanjean & Tanaka [6] and Shatah [9] have given a positive answer to this question, showing that

$$\min_{u \in \mathcal{S}_0} I_0(u) = \min_{u \in \mathcal{P}_0} I_0(u). \quad (7)$$

Inspired by these papers and observed that each solution of (1) satisfies the following Pohozaev identity:

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} V(x)u^2 + \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x) \mid x)u^2 = N \int_{\mathbb{R}^N} G(u), \quad (8)$$

we indicate with  $\mathcal{P}$  the Pohozaev manifold related to (1):

$$\mathcal{P} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid u \text{ satisfies (8)}\} \quad (9)$$

and we wonder if there exists a minimizer for  $I|_{\mathcal{P}}$ .

Proceeding in analogy with [1], we get the following result

**Theorem 1.5.** *If we assume (g1-4), (V1), (V3) and*

(V5)  $(\nabla V(x) \mid x) \leq 0$ , for all  $x \in \mathbb{R}^N$ ;

(V6)  $NV(x) + (\nabla V(x) \mid x) \geq 0$  for all  $x \in \mathbb{R}^N$ , and the inequality is strict somewhere,

then  $b := \inf_{u \in \mathcal{P}} I(u)$  is not a critical level for the functional  $I$ .

Requiring something more, instead of (V6), we have a further information on the level  $b$ :

**Corollary 1.6.** *Assuming the same hypotheses of Theorem 1.5, if in (V6) we have the strict inequality, then the level  $b$  is not achieved as a minimum on the Pohozaev manifold  $\mathcal{P}$ .*

**Acknowledgement** The authors express their deep gratitude to Prof. L. Jeanjean for stimulating discussions and useful suggestions.

## NOTATION

- For any  $1 \leq s < +\infty$ ,  $L^s(\mathbb{R}^3)$  is the usual Lebesgue space endowed with the norm

$$\|u\|_s^s := \int_{\mathbb{R}^3} |u|^s;$$

- $H^1(\mathbb{R}^N)$  is the usual Sobolev space endowed with the norm

$$\|u\|^2 := \int_{\mathbb{R}^3} |\nabla u|^2 + u^2;$$

- $\mathcal{D}^{1,2}(\mathbb{R}^N)$  is completion of  $C_0^\infty(\mathbb{R}^3)$  with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2 := \int_{\mathbb{R}^3} |\nabla u|^2;$$

- for any  $r > 0$ ,  $x \in \mathbb{R}^3$  and  $A \subset \mathbb{R}^3$

$$B_r(x) := \{y \in \mathbb{R}^3 \mid |y - x| \leq r\},$$

$$B_r := \{y \in \mathbb{R}^3 \mid |y| \leq r\},$$

$$A^c := \mathbb{R}^3 \setminus A.$$

## 2 The existence result

The aim of this section is to prove Theorems 1.1 and 1.2.

Set

$$H_r^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) \mid u \text{ is radial}\}$$

and, following [2], define  $s_0 := \min\{s \in [\zeta, +\infty[ \mid g(s) = 0\}$  ( $s_0 = +\infty$  if  $g(s) \neq 0$  for any  $s \geq \zeta$ ) and set  $\tilde{g} : \mathbb{R} \rightarrow \mathbb{R}$  the function such that

$$\tilde{g}(s) = \begin{cases} g(s) & \text{on } [0, s_0]; \\ 0 & \text{on } \mathbb{R}_+ \setminus [0, s_0]; \\ -\tilde{g}(-s) & \text{on } \mathbb{R}_-. \end{cases} \quad (10)$$

By the strong maximum principle, a solution of (1) with  $\tilde{g}$  in the place of  $g$  is a solution of (1). So we can suppose that  $g$  is defined as in (10), so that **(g1)**, **(g2)**, **(g4)** and then the following limit

$$\lim_{s \rightarrow \infty} \frac{|g(s)|}{|s|^{2^*-1}} = 0 \quad (11)$$

hold. Moreover, we set for any  $s \geq 0$ ,

$$\begin{aligned} g_1(s) &:= (g(s) + ms)^+, \\ g_2(s) &:= g_1(s) - g(s), \end{aligned}$$

and we extend them as odd functions.

Since

$$\lim_{s \rightarrow 0} \frac{g_1(s)}{s} = 0, \quad (12)$$

$$\lim_{s \rightarrow \infty} \frac{g_1(s)}{|s|^{2^*-1}} = 0, \quad (13)$$

and

$$g_2(s) \geq ms, \quad \forall s \geq 0, \quad (14)$$

by some computations, we have that for any  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$  such that

$$g_1(s) \leq C_\varepsilon s^{2^*-1} + \varepsilon g_2(s), \quad \forall s \geq 0. \quad (15)$$

If we set

$$G_i(t) := \int_0^t g_i(s) ds, \quad i = 1, 2,$$

then, by (14) and (15), we have

$$G_2(s) \geq \frac{m}{2} s^2, \quad \forall s \in \mathbb{R} \quad (16)$$

and for any  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$  such that

$$G_1(s) \leq \frac{C_\varepsilon}{2^*} |s|^{2^*} + \varepsilon G_2(s), \quad \forall s \in \mathbb{R}. \quad (17)$$

Using an idea from [4], we look for bounded Palais-Smale sequences of the following perturbed functionals

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 + \int_{\mathbb{R}^N} G_2(u) - \lambda \int_{\mathbb{R}^N} G_1(u),$$

for almost all  $\lambda$  near 1. Then we will deduce the existence of a non-trivial critical point  $v_\lambda$  of the functional  $I_\lambda$  at the mountain pass level. Afterward, we study the convergence of the sequence  $(v_\lambda)_{\lambda'}$ , as  $\lambda$  goes to 1 (observe that  $I_1 = I$ ).

We will apply the following slight modified version of [4, Theorem 1.1] (see [5]).

**Theorem 2.1.** *Let  $(X, \|\cdot\|)$  be a Banach space and  $J \subset \mathbb{R}^+$  an interval. Consider the family of  $C^1$  functionals on  $X$*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in J,$$

*with  $B$  nonnegative and either  $A(u) \rightarrow +\infty$  or  $B(u) \rightarrow +\infty$  as  $\|u\| \rightarrow \infty$ . For any  $\lambda \in J$  we set*

$$\Gamma_\lambda := \{\gamma \in C([0, 1], X) \mid \gamma(0) = 0 \neq \gamma(1), I_\lambda(\gamma(1)) < 0, \}. \quad (18)$$

*If for every  $\lambda \in J$  the set  $\Gamma_\lambda$  is nonempty and*

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0, 1]} I_\lambda(\gamma(t)) > I_\lambda(v), \quad (19)$$

*then for almost every  $\lambda \in J$  there is a sequence  $(v_n)_n \subset X$  such that*

- (i)  $(v_n)_n$  is bounded;
- (ii)  $I_\lambda(v_n) \rightarrow c_\lambda$ ;
- (iii)  $(I_\lambda)'(v_n) \rightarrow 0$  in the dual  $X^{-1}$  of  $X$ .

In our case,  $X = H_r^1(\mathbb{R}^N)$  and

$$\begin{aligned} A(u) &:= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 + \int_{\mathbb{R}^N} G_2(u), \\ B(u) &:= \int_{\mathbb{R}^N} G_1(u). \end{aligned}$$

In order to apply Theorem 2.1, we have just to define a suitable interval  $J$  such that  $\Gamma_\lambda \neq \emptyset$ , for any  $\lambda \in J$ , and (19) holds.

Observe that, according to [2], there exists a function  $z \in H_r^1(\mathbb{R}^N)$  such that

$$\int_{\mathbb{R}^N} G_1(z) - \int_{\mathbb{R}^N} G_2(z) = \int_{\mathbb{R}^N} G(z) > 0.$$

Then there exists  $0 < \bar{\delta} < 1$  such that

$$\bar{\delta} \int_{\mathbb{R}^N} G_1(z) - \int_{\mathbb{R}^N} G_2(z) > 0. \quad (20)$$

We define  $J$  as the interval  $[\bar{\delta}, 1]$ .

**Lemma 2.2.**  $\Gamma_\lambda \neq \emptyset$ , for any  $\lambda \in J$ .

**Proof** Let  $\lambda \in J$ . Set  $\bar{\theta} > 0$  sufficiently large and  $\bar{z} = z(\cdot/\bar{\theta})$ . Define  $\gamma : [0, 1] \rightarrow H_r^1(\mathbb{R}^N)$  in the following way

$$\gamma(t) = \begin{cases} 0, & \text{if } t = 0, \\ \bar{z}^t = \bar{z}(\cdot/t), & \text{if } 0 < t \leq 1. \end{cases}$$

It is easy to see that  $\gamma$  is a continuous path from 0 to  $\bar{z}$ . Moreover, we have that

$$\begin{aligned} I_\lambda(\gamma(1)) &\leq \frac{\bar{\theta}^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla z|^2 + \frac{\bar{\theta}^N}{2} \int_{\mathbb{R}^N} V(\bar{\theta}x) z^2 \\ &\quad + \bar{\theta}^N \left( \int_{\mathbb{R}^N} G_2(z) - \bar{\delta} \int_{\mathbb{R}^N} G_1(z) \right) \end{aligned}$$

and then, by (20), (V3) and the Lebesgue theorem, for a suitable choice of  $\bar{\theta}$ , certainly  $\gamma \in \Gamma_\lambda$ .  $\square$

**Lemma 2.3.**  $c_\lambda > 0$  for all  $\lambda \in J$ .

**Proof** Observe that for any  $u \in H_r^1(\mathbb{R}^N)$  and  $\lambda \in J$ , using (16) and (17) for  $\varepsilon < 1$ , we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 + \int_{\mathbb{R}^N} G_2(u) - \int_{\mathbb{R}^N} G_1(u) \\ &\geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + (1 - \varepsilon) \frac{m}{2} \int_{\mathbb{R}^N} u^2 - \frac{C_\varepsilon}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} \end{aligned}$$



and then, by Sobolev embeddings, we conclude that there exists  $\rho > 0$  such that for any  $\lambda \in J$  and  $u \in H_r^1(\mathbb{R}^N)$  with  $u \neq 0$  and  $\|u\| \leq \rho$ , it results  $I_\lambda(u) > 0$ . In particular, for any  $\|u\| = \rho$ , we have  $I_\lambda(u) \geq \tilde{c} > 0$ . Now fix  $\lambda \in J$  and  $\gamma \in \Gamma_\lambda$ . Since  $\gamma(0) = 0$  and  $I_\lambda(\gamma(1)) < 0$ , certainly  $\|\gamma(1)\| > \rho$ . By continuity, we deduce that there exists  $t_\gamma \in ]0, 1[$  such that  $\|\gamma(t_\gamma)\| = \rho$ . Therefore, for any  $\lambda \in J$ ,

$$c_\lambda \geq \inf_{\gamma \in \Gamma_\lambda} I_\lambda(\gamma(t_\gamma)) \geq \tilde{c} > 0. \quad (21)$$

□

We present a variant of the Strauss' compactness lemma [10] (see also [2, Theorem A.1]), whose proof is similar to that contained in [2]. It will be a fundamental tool in our arguments:

**Lemma 2.4.** *Let  $P$  and  $Q : \mathbb{R} \rightarrow \mathbb{R}$  be two continuous functions satisfying*

$$\lim_{s \rightarrow \infty} \frac{P(s)}{Q(s)} = 0,$$

*$(v_n)_n$ ,  $v$  and  $z$  be measurable functions from  $\mathbb{R}^N$  to  $\mathbb{R}$  such that*

$$\begin{aligned} \sup_n \int_{\mathbb{R}^N} |Q(v_n(x))z| dx &< +\infty, \\ P(v_n(x)) &\rightarrow v(x) \text{ a.e. in } \mathbb{R}^N. \end{aligned}$$

*Then  $\|(P(v_n) - v)z\|_{L^1(B)} \rightarrow 0$ , for any bounded Borel set  $B$ .*

*Moreover, if we have also*

$$\begin{aligned} \lim_{s \rightarrow 0} \frac{P(s)}{Q(s)} &= 0, \\ \lim_{x \rightarrow \infty} \sup_n |v_n(x)| &= 0, \end{aligned}$$

*then  $\|(P(v_n) - v)z\|_{L^1(\mathbb{R}^N)} \rightarrow 0$ .*

In analogy with the well-known compactness result in [3], we state the following result

**Lemma 2.5.** *For any  $\lambda \in J$ , each bounded Palais-Smale sequence for the functional  $I_\lambda$  admits a convergent subsequence.*

**Proof** Let  $\lambda \in J$  and  $(u_n)_n$  be a bounded (PS) sequence for  $I_\lambda$ , namely

$$\begin{aligned} (I_\lambda(u_n))_n &\text{ is bounded,} \\ \lim_n (I_\lambda)'(u_n) &= 0 \text{ in } (H_r^1(\mathbb{R}^N))'. \end{aligned} \quad (22)$$

Up to a subsequence, we can suppose that there exists  $u \in H_r^1(\mathbb{R}^N)$  such that

$$u_n \rightharpoonup u \text{ weakly in } H_r^1(\mathbb{R}^N) \quad (23)$$

and

$$u_n(x) \rightarrow u(x) \text{ a.e. in } \mathbb{R}^N. \quad (24)$$

By weak lower semicontinuity we have:

$$\int_{\mathbb{R}^N} |\nabla u|^2 \leq \liminf_n \int_{\mathbb{R}^N} |\nabla u_n|^2; \quad (25)$$

$$\int_{\mathbb{R}^N} V(x)u^2 \leq \liminf_n \int_{\mathbb{R}^N} V(x)u_n^2. \quad (26)$$

If we apply Lemma 2.4 for  $P(s) = g_i(s)$ ,  $i = 1, 2$ ,  $Q(s) = |s|^{2^*-1}$ ,  $(v_n)_n = (u_n)_n$ ,  $v = g_i(u)$ ,  $i = 1, 2$  and  $z \in C_0^\infty(\mathbb{R}^N)$ , by (11), (13) and (24) we deduce that

$$\int_{\mathbb{R}^N} g_i(u_n)z \rightarrow \int_{\mathbb{R}^N} g_i(u)z \quad i = 1, 2.$$

As a consequence, by (22) and (23) we deduce  $(I_\lambda)'(u) = 0$  and hence

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 = \int_{\mathbb{R}^N} \lambda g_1(u)u - g_2(u)u. \quad (27)$$

If we apply Lemma 2.4 for  $P(s) = g_1(s)s$ ,  $Q(s) = s^2 + |s|^{2^*}$ ,  $(v_n)_n = (u_n)_n$ ,  $v = g_1(u)u$ , and  $z = 1$ , by (11), (13), (24) and the well known Strauss' radial lemma (see [10]) we deduce that

$$\int_{\mathbb{R}^N} g_1(u_n)u_n \rightarrow \int_{\mathbb{R}^N} g_1(u)u. \quad (28)$$

Moreover, by (24) and Fatou's lemma

$$\int_{\mathbb{R}^N} g_2(u)u \leq \liminf_n \int_{\mathbb{R}^N} g_2(u_n)u_n. \quad (29)$$

By (27), (28) and (29), and since  $\langle (I_\lambda)'(u_n), u_n \rangle \rightarrow 0$

$$\begin{aligned} \limsup_n \int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)u_n^2 &= \limsup_n \left[ \lambda \int_{\mathbb{R}^N} g_1(u_n)u_n - \int_{\mathbb{R}^N} g_2(u_n)u_n \right] \\ &\leq \lambda \int_{\mathbb{R}^N} g_1(u)u - \int_{\mathbb{R}^N} g_2(u)u \\ &= \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2. \end{aligned} \quad (30)$$

By (25), (26) and (30), we get

$$\begin{aligned} \lim_n \int_{\mathbb{R}^N} |\nabla u_n|^2 &= \int_{\mathbb{R}^N} |\nabla u|^2, \\ \lim_n \int_{\mathbb{R}^N} V(x) u_n^2 &= \int_{\mathbb{R}^N} V(x) u^2, \end{aligned} \quad (31)$$

hence

$$\lim_n \int_{\mathbb{R}^N} g_2(u_n) u_n = \int_{\mathbb{R}^N} g_2(u) u. \quad (32)$$

Since  $g_2(s)s = ms^2 + q(s)$ , with  $q$  a positive and continuous function, by Fatou's Lemma we have

$$\begin{aligned} \int_{\mathbb{R}^N} q(u) &\leq \liminf_n \int_{\mathbb{R}^N} q(u_n); \\ \int_{\mathbb{R}^N} u^2 &\leq \liminf_n \int_{\mathbb{R}^N} u_n^2. \end{aligned}$$

These last two inequalities and (32) imply that, up to a subsequence,

$$\int_{\mathbb{R}^N} u^2 = \lim_n \int_{\mathbb{R}^N} u_n^2,$$

which, together with (31), shows that  $u_n \rightarrow u$  strongly in  $H_r^1(\mathbb{R}^N)$ .  $\square$

**Lemma 2.6.** *For almost every  $\lambda \in J$ , there exists  $u^\lambda \in H_r^1(\mathbb{R}^N)$ ,  $u^\lambda \neq 0$ , such that  $(I_\lambda)'(u^\lambda) = 0$  and  $I_\lambda(u^\lambda) = c_\lambda$ .*

**Proof** By Theorem 2.1, for almost every  $\lambda \in J$ , there exists a bounded sequence  $(u_n^\lambda)_n \subset H_r^1(\mathbb{R}^N)$  such that

$$I_\lambda(u_n^\lambda) \rightarrow c_\lambda; \quad (33)$$

$$(I_\lambda)'(u_n^\lambda) \rightarrow 0 \text{ in } (H_r^1(\mathbb{R}^N))'. \quad (34)$$

Up to a subsequence, by Lemma 2.5, we can suppose that there exists  $u^\lambda \in H_r^1(\mathbb{R}^N)$  such that  $u_n^\lambda \rightarrow u^\lambda$  in  $H_r^1(\mathbb{R}^N)$ . By Lemma 2.3, (33) and (34) we conclude.  $\square$

Now we are able to provide the proof of our main result:

**Proof of Theorem 1.1** By Lemma 2.6, we are allowed to consider a suitable  $\lambda_n \nearrow 1$  such that for any  $n \geq 1$  there exists  $v_n \in H_r^1(\mathbb{R}^N) \setminus \{0\}$  satisfying

$$I_{\lambda_n}(v_n) = c_{\lambda_n}, \quad (35)$$

$$(I_{\lambda_n})'(v_n) = 0 \text{ in } (H_r^1(\mathbb{R}^N))'. \quad (36)$$

We want to prove that  $(v_n)_n$  is a bounded Palais-Smale sequence for  $I$  at the level  $c := c_1$ .

By standard argument, since  $H_r^1(\mathbb{R}^N)$  is a natural constraint, we have that  $v_n$  is a weak solution of the problem

$$-\Delta w + V(x)w + g_2(w) - \lambda_n g_1(w) = 0$$

and it satisfies the Pohozaev equality

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v_n|^2 + \frac{N}{N-2} \int_{\mathbb{R}^N} V(x)v_n^2 + \frac{1}{N-2} \int_{\mathbb{R}^N} (\nabla V(x) \mid x)v_n^2 \\ + \frac{2N}{N-2} \int_{\mathbb{R}^N} G_2(v_n) - \lambda_n G_1(v_n) = 0. \end{aligned} \quad (37)$$

Therefore, by (35), (36) and (37) we have that the following system holds

$$\begin{cases} \frac{1}{2}(\alpha_n + \beta_n) + \gamma_{2,n} - \lambda_n \gamma_{1,n} = c_{\lambda_n}, \\ \alpha_n + \beta_n + \delta_{2,n} - \lambda_n \delta_{1,n} = 0, \\ \alpha_n + \frac{N}{N-2}\beta_n + \frac{1}{N-2}\eta_n + \frac{2N}{N-2}\gamma_{2,n} - \frac{2N}{N-2}\lambda_n \gamma_{1,n} = 0, \end{cases} \quad (38)$$

where

$$\begin{aligned} \alpha_n &= \int_{\mathbb{R}^N} |\nabla v_n|^2, \quad \beta_n = \int_{\mathbb{R}^N} V(x)v_n^2, \quad \eta_n = \int_{\mathbb{R}^N} (\nabla V(x) \mid x)v_n^2, \\ \gamma_{i,n} &= \int_{\mathbb{R}^N} G_i(v_n), \quad \delta_{i,n} = \int_{\mathbb{R}^N} g_i(v_n)v_n, \quad i = 1, 2. \end{aligned}$$

By the first and the third of the system we get

$$\frac{1}{N}\alpha_n - \frac{1}{2N}\eta_n = c_{\lambda_n}$$

and then, using Holder inequality, by (V2) and the boundedness of  $(c_{\lambda_n})_n$  (indeed the map  $\lambda \mapsto c_\lambda$  is non-increasing), we have

$$\alpha_n \leq C \text{ for all } n \geq 1. \quad (39)$$

By the second of the system we have

$$\delta_{2,n} - \lambda_n \delta_{1,n} = -\alpha_n - \beta_n \leq 0$$

and then by (15), there exists  $0 < \varepsilon < 1$  and  $C_\varepsilon > 0$  such that

$$\delta_{2,n} \leq \delta_{1,n} \leq C_\varepsilon \int_{\mathbb{R}^N} |v_n|^{2^*} + \varepsilon \delta_{2,n}.$$

Therefore, by the Sobolev embedding  $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$

$$(1 - \varepsilon) \delta_{2,n} \leq C_\varepsilon \int_{\mathbb{R}^N} |v_n|^{2^*} \leq C \alpha_n^{2^*}$$

and then, by (39),  $\delta_{2,n}$  is bounded. By (14) and (39) we deduce that  $(v_n)_n$  is bounded in  $H_r^1(\mathbb{R}^N)$ .

Up to a subsequence, there exists  $v \in H_r^1(\mathbb{R}^N)$  such that

$$v_n \rightharpoonup v \text{ weakly in } H_r^1(\mathbb{R}^N). \quad (40)$$

By (36), we have that

$$I'(v_n) = (I_{\lambda_n})'(v_n) + (\lambda_n - 1)g_1(v_n) = (\lambda_n - 1)g_1(v_n)$$

so

$$I'(v_n) \rightarrow 0 \quad \text{in } (H_r^1(\mathbb{R}^N))', \quad (41)$$

if  $(g_1(v_n))_n$  is bounded in  $(H_r^1(\mathbb{R}^N))'$ . But this is true by the Banach-Steinhaus theorem, since Lemma 2.4 implies that for any  $z \in H_r^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} g_1(v_n)z \rightarrow \int_{\mathbb{R}^N} g_1(v)z.$$

Moreover, from (35) and the boundedness of  $(v_n)_n$ , we deduce that

$$I(v_n) = I_{\lambda_n}(v_n) + (\lambda_n - 1) \int_{\mathbb{R}^N} G_1(v_n) \rightarrow c. \quad (42)$$

Therefore, by (41) and (42),  $(v_n)_n$  is a Palais-Smale sequence for the functional  $I$  and so, by Lemma 2.5,  $v$  is a nontrivial mountain pass type solution for (1).

To conclude, observe that, by standard arguments, we can use the strong maximum principle to get  $v > 0$ .

□

In order to prove Theorem 1.2 we set

$$\begin{aligned} \mathcal{S}_r &:= \{u \in H_r^1(\mathbb{R}^N) \setminus \{0\} \mid I'(u) = 0\}, \\ \sigma_r &:= \inf_{u \in \mathcal{S}_r} I(u). \end{aligned}$$

**Lemma 2.7.** *We have  $\sigma_r > 0$ .*

**Proof** By (15) and Sobolev embedding we have that for any  $u \in \mathcal{S}_r$

$$\begin{aligned} \|\nabla u\|_2^2 &\leq \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 + (1 - \varepsilon) \int_{\mathbb{R}^N} g_2(u)u \\ &\leq C_\varepsilon \|u\|_{2^*}^{2^*} \leq C \|\nabla u\|_2^{2^*} \end{aligned}$$

where  $0 < \varepsilon < 1$  and  $C_\varepsilon, C > 0$ . So we deduce that

$$\inf_{u \in \mathcal{S}_r} \|\nabla u\|_2 > 0.$$

Now, since  $\mathcal{S}_r \subset \mathcal{P}$  (see (9)), for any  $u \in \mathcal{S}_r$  by **(V2)** we have

$$I(u) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2N} \int_{\mathbb{R}^N} (\nabla V(x) \mid x) u^2 \geq C > 0.$$

□

Finally we provide the following

**Proof of Theorem 1.2** Let  $(u_n)_n \in \mathcal{S}_r$  such that  $I(u_n) \rightarrow \sigma_r$ . Arguing as in the proof of Theorem 1.1 we have that the sequence is bounded. By Lemma 2.5, there exists  $u \in H_r^1(\mathbb{R}^N)$  such that  $u_n \rightarrow u$  in  $H_r^1(\mathbb{R}^N)$  and then the conclusion follows by Lemma 2.7. □

### 3 The nonexistence result

In this section we give the proof of Theorem 1.5 and we will assume that  $V$  satisfies hypotheses **(V1)**, **(V3)**, **(V5-6)**.

Let us show that the functional  $I$  is bounded below on the manifold  $\mathcal{P}$ :

**Lemma 3.1.** *For all  $u \in \mathcal{P}$ ,  $I(u) > 0$ .*

**Proof** It is easy to see that for any  $u \in \mathcal{P}$ , by **(V2)** we get

$$I(u) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2N} \int_{\mathbb{R}^N} (\nabla V(x) \mid x) u^2 > 0. \quad (43)$$

□

By Lemma 3.1, we can define

$$b = \inf_{u \in \mathcal{P}} I(u) \geq 0. \quad (44)$$

**Lemma 3.2.** *Let  $w \in H^1(\mathbb{R}^N)$  be such that  $\int_{\mathbb{R}^N} G(w) > 0$ . Then there exists  $\bar{\theta} > 0$  such that  $w^{\bar{\theta}} = w(\cdot/\bar{\theta}) \in \mathcal{P}$ . In particular this result is true for any  $w \in \mathcal{P}_0$  (see (6)).*

**Proof** For any  $\theta > 0$ , we set

$$f(\theta) := I(w^\theta) = \frac{\theta^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + \frac{\theta^N}{2} \int_{\mathbb{R}^N} V(\theta x) w^2 - \theta^N \int_{\mathbb{R}^N} G(w).$$

By the Lebesgue theorem and (V3), we get

$$\lim_{\theta \rightarrow +\infty} \int_{\mathbb{R}^N} V(\theta x) w^2 = 0,$$

and so

$$\lim_{\theta \rightarrow +\infty} f(\theta) = -\infty.$$

We argue that there exists  $\bar{\theta} > 0$  such that  $f'(\bar{\theta}) = 0$ : hence  $w^{\bar{\theta}} \in \mathcal{P}$ .  $\square$

Let  $w \in \mathcal{P}_0$ . For any  $y \in \mathbb{R}^N$ , we set  $w_y := w(\cdot - y) \in \mathcal{P}_0$ . Set  $\theta_y > 0$  such that  $\tilde{w}_y = w_y(\cdot/\theta_y) \in \mathcal{P}$ .

**Lemma 3.3.** *We have  $\lim_{|y| \rightarrow \infty} \theta_y = 1$ .*

**Proof** STEP 1:  $\limsup_{|y| \rightarrow \infty} \theta_y < +\infty$ .

Suppose, by contradiction, that  $\theta_{y_n} \rightarrow +\infty$ , for  $|y_n| \rightarrow \infty$ .

For any  $y \in \mathbb{R}^N$ , we have

$$I(\tilde{w}_y) = \frac{\theta_y^{N-2}}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + \frac{\theta_y^N}{2} \int_{\mathbb{R}^N} V(\theta_y x) w^2 \left( x - \frac{y}{\theta_y} \right) - \theta_y^N \int_{\mathbb{R}^N} G(w). \quad (45)$$

Let us show that

$$\lim_{|y| \rightarrow \infty} \int_{\mathbb{R}^N} V(\theta_y x) w^2 \left( x - \frac{y}{\theta_y} \right) = 0. \quad (46)$$

Indeed we have

$$\begin{aligned} \int_{\mathbb{R}^N} V(\theta_y x) w^2 \left( x - \frac{y}{\theta_y} \right) &= \int_{B_r} V(\theta_y x) w^2 \left( x - \frac{y}{\theta_y} \right) + \int_{B_r^c} V(\theta_y x) w^2 \left( x - \frac{y}{\theta_y} \right) \\ &\leq \sup_{x \in \mathbb{R}^N} V(x) \int_{B_r(-y/\theta_y)} w^2 + \sup_{x \in B_r^c} V(\theta_y x) \|w\|_2^2. \end{aligned} \quad (47)$$

By the absolute continuity of the Lebesgue integral, for any  $\varepsilon > 0$  there exists  $\bar{r} > 0$  such that, for any  $r < \bar{r}$  and for any  $y \in \mathbb{R}^N$ , we get

$$\sup_{x \in \mathbb{R}^N} V(x) \cdot \int_{B_r(-y/\theta_y)} w^2 \leq \varepsilon. \quad (48)$$

Therefore, since we are supposing that  $\theta_{y_n} \rightarrow +\infty$ , as  $|y_n| \rightarrow \infty$ , by (47), (48) and (V2), we get (46). As a consequence, by (45) and (46), we infer that  $I(\tilde{w}_{y_n}) \rightarrow -\infty$ , as  $|y_n| \rightarrow \infty$ , and we get a contradiction with Lemma 3.1.

STEP 2:  $\lim_{|y| \rightarrow \infty} \theta_y = 1$ .

Since  $w \in \mathcal{P}_0$  and  $\tilde{w}_y \in \mathcal{P}$ , we get

$$\begin{aligned} N(\theta_y^2 - 1) \int_{\mathbb{R}^N} G(w) \\ = \frac{\theta_y^2}{2} \int_{\mathbb{R}^N} [NV(\theta_y x + y) + (\nabla V(\theta_y x + y) \mid (\theta_y x + y))] w^2. \end{aligned} \quad (49)$$

By (V3), (V5) and (V6), using the dominated convergence and the conclusion of the Step 1, the right hand side in (49) goes to zero as  $|y| \rightarrow \infty$ , and so the lemma is proved.  $\square$

We set (see (7))

$$b_0 := \min_{u \in \mathcal{S}_0} I_0(u) = \min_{u \in \mathcal{P}_0} I_0(u).$$

**Lemma 3.4.**  $b \leq b_0$ .

**Proof** Let  $w \in H^1(\mathbb{R}^N)$  be a ground state solution of (3). Then  $w \in \mathcal{P}_0$  and  $I_0(w) = b_0$ . For any  $y \in \mathbb{R}^N$ , we set  $w_y = w(\cdot - y)$ . By the invariance by translations of (3), we have that  $w_y \in \mathcal{P}_0$  and  $I_0(w_y) = b_0$ . By Lemma 3.2, for any  $y \in \mathbb{R}^N$  there exists  $\theta_y > 0$  such that  $\tilde{w}_y = w_y(\cdot/\theta_y) \in \mathcal{P}$ . We get

$$\begin{aligned} |I(\tilde{w}_y) - b_0| &= |I(\tilde{w}_y) - I_0(w_y)| \\ &\leq \frac{|\theta_y^{N-2} - 1|}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + \frac{\theta_y^N}{2} \int_{\mathbb{R}^N} V(\theta_y x + y) w^2 \\ &\quad + |\theta_y^N - 1| \int_{\mathbb{R}^N} G(w). \end{aligned}$$

Therefore, by Lemma 3.3, we infer that

$$\lim_{|y| \rightarrow \infty} I(\tilde{w}_y) = b_0,$$



hence  $b \leq b_0$ .  $\square$

**Lemma 3.5.** *Let  $z \in H^1(\mathbb{R}^N)$  be such that  $\int_{\mathbb{R}^N} G(z) > 0$ . Then there exists  $\bar{\theta} > 0$  such that  $z^{\bar{\theta}} = z(\cdot/\bar{\theta}) \in \mathcal{P}_0$ . In particular, by (V6) this result is true for any  $z \in \mathcal{P}$  with  $\bar{\theta} \leq 1$ .*

**Proof** The first part of the statement follows from the fact that, for any  $z \in H^1(\mathbb{R}^N)$  such that  $\int_{\mathbb{R}^N} G(z) > 0$ , certainly there exists  $\bar{\theta} > 0$  such that

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla z|^2 = N\bar{\theta}^2 \int_{\mathbb{R}^N} G(z). \quad (50)$$

Consider now the case of  $z \in \mathcal{P}$ . Since

$$\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla z|^2 + \frac{N}{2} \int_{\mathbb{R}^N} V(x)z^2 + \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x) \mid x)z^2 = N \int_{\mathbb{R}^N} G(z), \quad (51)$$

by (V6) we have  $\int_{\mathbb{R}^N} G(z) > 0$ . Let  $\bar{\theta} > 0$  such that (50) holds. Combining (50) and (51) we get

$$\frac{1}{2} \int_{\mathbb{R}^N} [NV(x) + (\nabla V(x) \mid x)]z^2 = N(1 - \bar{\theta}^2) \int_{\mathbb{R}^N} G(z). \quad (52)$$

By (V6), we get the conclusion.  $\square$

Now we can prove Theorem 1.5:

**Proof of Theorem 1.5** Suppose by contradiction that there exists  $z \in H^1(\mathbb{R}^N)$  critical point of the functional  $I$  at level  $b$ : in particular,  $z \in \mathcal{P}$  and  $I(z) = b$ . Let  $\theta \in (0, 1]$  be such that  $z^\theta \in \mathcal{P}_0$ . Let us show that  $\theta < 1$ .

By standard arguments and using the strong maximum principle, we infer that  $z$  does not change sign and so we can assume that  $z > 0$ . Therefore, by (V6) and (52), we get that  $\theta < 1$ .

By (43) and (V5), we infer that

$$\begin{aligned} b = I(z) &= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla z|^2 - \frac{1}{2N} \int_{\mathbb{R}^N} (\nabla V(x) \mid x)z^2 \\ &> \frac{\theta^{N-2}}{N} \int_{\mathbb{R}^N} |\nabla z|^2 = I_0(z^\theta) \geq b_0, \end{aligned}$$

and we get a contradiction with Lemma 3.4.  $\square$

**Proof of Corollary 1.6** If the strict inequality in (V6) is satisfied almost everywhere, then for any  $z \in \mathcal{P}$  there exists  $\theta \in (0, 1)$  such that  $z^\theta \in \mathcal{P}_0$ . Arguing as in the proof of Theorem 1.5 we conclude.  $\square$

**Remark 3.6.** *In view of Theorem 1.5, the proof of Corollary 1.6 would follow immediately if  $\mathcal{P}$  was a natural constraint for the functional  $I$ .*

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